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ON STEADY ROTATIONS OF A RIGID BODY IN A PERIODIC ORBIT NEAR THE COLLINEAR LIBRATION POINT

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A. P. MARKEEV
(Moscow)

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The paper deals with the problem of motion of a dynamically symmetric rigid body about the center of mass, near the collinear libration point L_2 of the bounded circular three-body (material points) problem. It is assumed that the periodic orbit of the center of mass of the rigid body represents a segment of a straight line, perpendicular to the plane of rotation of the principal attractive masses and passing through L_2. Two types or rotation of the rigid body stationary with respect to the orbital coordinate system are found, and their stability studied in the first approximation.

1. Formulation of the problem. A rotational motion of a rigid body the center of mass of which moves along a periodic orbit near the collinear libration point L_2, takes place under the action of gravitational moments depending on the material points m_1 and m_2. The linear dimensions of the body are small compared with the distances separating its center of mass O from the points m_1 and m_2, therefore we assume that the motion of the rigid body relative to its center of mass does not affect the motion of the center of mass itself. We shall also assume that the orbit of the center of mass O of the rigid body is defined within the framework of the bounded circular three-body (or more accurately three-point m_1, m_2 and O) problem.

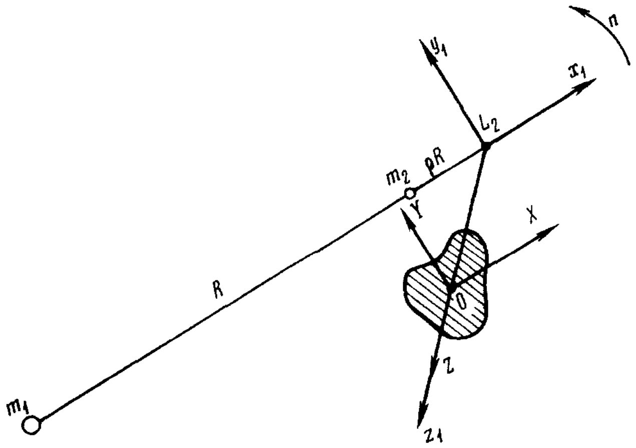

Fig. 1.

Figure 1 depicts the $L_2x_1y_1z_1$ -coordinate system. Its L_2x_1 -axis is directed along the line m_1m_2, the L_2y_1 -axis is situated in the plane of motions of the principal attracting masses m_1 and m_2 and points in the direction of their rotation, and L_2z_1 -axis complements the L_2x_1 and L_2y_1 axes to form a right coordinate system. The arrow shows the direction of rotation of m_1 and m_2, and n denotes the angular velocity of rotation of m_1 and m_2. For the system Earth—Moon $n =$ 0.23 radians/24 hours, and this corresponds to the period of rotation of the Moon about the Earth (sidereal month) equal to 27.3 days. R denotes the distance between the points m_1 and m_2, and ρR the distance between m_2 and L_2. The quantity ρ is a root of a fifth degree polynomial with the coefficients depending on μ. For the system Earth—Moon, $\mu = m_2 / (m_1 + m_2) = 0.01215$ and $\rho = 0.1678$.

In the bounded, circular problem of three material points where the coordinate system rotates together with m_1 and m_2, there exist near L_2 two, two-parameter families of periodic motions of a point of infinitesimal mass [1] (in the present case the point is represented by the center of mass O of the rigid body). If we neglect in the equations of motion of the bounded problem of three points the nonlinear terms with respect to the deviations from L_2, then the trajectory of one of the periodic motions above will represent a straight line segment perpendicular to the plane of rotation of the points m_1 and m_2 and passing through L_2. Let us write the equation of this trajectory in the form

$$x_1 \equiv 0, \ y_1 \equiv 0, \ z_1 = \varepsilon R \sin \omega_z nt \tag{1.1}$$

$$\omega_z = \sqrt{A_2}, \quad A_2 = \frac{1-\mu}{(1+\rho)^3} + \frac{\mu}{\rho^3}$$

where ε is an arbitrary small parameter representing the constant of integration. The other constant of integration is assumed to be zero.

For the system Earth —Moon we have $\omega_z = 1.786$, and this corresponds to the period of motion of a point with infinitesimal mass along the orbit (1.1), equal to 15.3 days.

We shall assume that the center of mass O of the rigid body moves along the orbit (1.1) and we neglect in all equations terms of the order higher than one in ε.

It should be noted that the periodic motions of a point with infinitesimal mass are unstable near L_2. In spite of this, the problem of motion of a rigid body relative to the center of mass formulated here is of interest not only from the theoretical point of view, but also in practice. The motion of the center of mass of a rigid body along the unstable periodic orbit (1.1) could be maintained e.g. by a controlling acceleration the vector of which would pass through the center of the body. Diverse practical examples using the libration points of the three-body problem are given in e.g. [2].

2. **Equations of motion of a rigid body relative to its center of mass.** The figure shows the orbital $OXYZ$ -coordinate system. The system has its origin at the center of mass of the rigid body which remains, by definition, on the L_2z_1 -axis at all times. The directions of the OX, OY and OZ axes coincide with the directions of the L_2x_1, L_2y_1 and L_2z_1 axes respecively. The Ox, Oy and Oz axes of the coordinate system associated with the rigid body and not shown in Figure, coincide with the principal axes of inertia of the body, and the Oz -axis is directed along the dynamic symmetry axis

of the body. The angles of precession ψ, nutation θ and characteristic rotation φ define the relative orientation of the associated and the orbital coordinate systems.

We shall write the equations of motion of the rigid body relative to its center of mass in the form of the Lagrange equations of second kind. Let A be the equatorial, and C the polar moment of inertia of the body. Then the kinetic energy of the body will be given by the formula

$$T = {}^1\!/_2\, A\,(p^2 + q^2) + {}^1\!/_2\, Cr^2 \qquad (2.1)$$

where p, q and r are the projections of the absolute angular velocity of the body onto the principal central axes of inertia Ox, Oy and Oz respectively. We note that by virtue of the dynamic symmetry of the body the quantity $r = (\psi^{\cdot} + n)\cos\theta + \varphi^{\cdot}$ will represent the integral of motion. Let us put $r = r_0 = \mathrm{const}$. The force function U can be written in the form

$$U = -\frac{3}{2}(C - A)\sum_{i=1}^{2}\frac{k_i}{r_i^3}\alpha_i^2 \qquad (2.2)$$

where α_i are the cosines of the angles formed by the directions of the radius vectors $\mathbf{r}_i$ of the center of mass of the body relative to the points m_i and the Oz-axis of the associated coordinate system, $k_i = fm_i$ and f is the universal gravitational constant. We have the following relations:

$$f\,(m_1 + m_2) = n^2R^3,\quad k_1 = (1 - \mu)\, n^2R^3,\quad k_2 = \mu n^2R^3$$

It can be shown that the equations

$$r_1 = (1 + \rho)\, R,\quad r_2 = \rho R \qquad (2.3)$$

$$\alpha_1 = \sin\psi\sin\theta + \frac{\varepsilon}{1+\rho}\cos\theta\sin\omega_z nt$$

$$\alpha_2 = \sin\psi\sin\theta + \frac{\varepsilon}{\rho}\cos\theta\sin\omega_z nt$$

hold to within the quantities of the order of ε. The relations (2.1)–(2.3) yield the Lagrange's function $L = T + U$ and hence the equations of motion of the rigid body relative to the center of mass. Omitting the standard manipulations, we give the equations in their final form

$$\begin{aligned} &\sin\theta\psi'' + 2\cos\theta\,(\psi' + 1)\,\theta' - ab\theta' + \\ &\quad 3\,(a - 1)\cos\psi\,(A_2\sin\psi\sin\theta + \varepsilon A_3\cos\theta\sin\omega_z\tau) = 0 \\ &\theta'' - \sin\theta\cos\theta\,(\psi' + 1)^2 + ab\sin\theta\,(\psi' + 1) + \\ &\quad 3\,(a - 1)\sin\psi\,(A_2\sin\psi\sin\theta\cos\theta + \\ &\quad \varepsilon A_3\cos 2\theta\sin\omega_z\tau) = 0 \\ &a = \frac{C}{A}\quad (0 \leqslant a \leqslant 2),\quad b = \frac{r_0}{n},\quad A_k = \frac{1-\mu}{(1+\rho)^{k+1}} + \frac{\mu}{\rho^{k+1}} \\ &(k = 2, 3) \end{aligned} \qquad (2.4)$$

where a prime denotes differentiation with respect to the independent variable $\tau = nt$.

3. **Two types of the steady rotations.** The positions of

equilibrium $\psi =$ const, $\theta =$ const of the system (2.4) (provided that they exist) correspond to steady rotations of the rigid body in the orbital coordinate system. The dynamic symmetry axis of the body occupies for these rotations a fixed position in the orbital coordinate system, and the rigid body itself rotates about the symmetry axis with a constant angular velocity of $\varphi^{\cdot} = (b - \cos\theta)\, n$.

Let us consider the problem of existence of the steady rotations. Putting $\psi' = \theta' = \psi'' = \theta'' \equiv 0$ in the equations of motion (2.4), we obtain a system of equations for determining the positions of equilibrium $\psi =$ const, $\theta =$ const (we assume that $a \neq 1$, i.e. that the inertia ellipsoid of the rigid body is not a sphere)

$$\begin{aligned} &\cos\psi\,(A_2 \sin\psi \sin\theta + \varepsilon A_3 \cos\theta \sin\omega_z\tau) = 0 \qquad (3.1)\\ &\sin\theta\,(ab - \cos\theta) + 3\,(a-1)\sin\psi\,(A_2 \sin\psi \sin\theta\cos\theta + \\ &\quad \varepsilon A_3 \cos 2\theta \sin\omega_z\tau) = 0 \end{aligned}$$

It is essential that the system (3.1) become an identity in τ when the angles ψ and θ are constant. From this we obtain two types of steady rotations of a rigid body.

A rotation of the first type exists only when $b = 0$ and the constant values of the angles ψ_0 and θ_0 satisfy the equations

$$\sin\psi_0 = 0, \ \cos\theta_0 = 0 \qquad (3.2)$$

For the motions (3.2) the Oz -axis of the associated coordinate system lies on the OY -axis of the orbital coordinate system. Consequently, the symmetry axis of the rigid body remains, during its whole motion, in a plane passing through L_2 and perpendicular to the line m_1m_2, and remains parallel to the plane of rotation of the points m_1 and m_2. At the same time, the rigid body does not rotate about its symmetry axis, and its center of mass executes a periodic motion along the normal to the plane containing the orbits of the points m_1 and m_2 and passing through L_2.

A steady rotation of the second type exists when the parameters a and b are connected by the relation

$$ab + [3\,(a-1)\,A_2 - 1]\cos\theta_0 = 0 \qquad (3.3)$$

and the constant values of the angles ψ_0 and θ_0 satisfy the equations

$$\cos\psi_0 = 0, \ \cos 2\theta_0 = 0 \qquad (3.4)$$

For the motions (3.4) the Oz -axis of the associated coordinate system is perpendicular to the OY-axis of the orbital system and is directed along the bisectrix of the angle XOZ. Consequently the symmetry axis of the rigid body lies, during the whole of its motion, in a plane passing through the points m_1 and m_2, is perpendicular to the plane of their rotation, and forms the angle of $\pi / 4$ with the latter.

In what follows, we shall assume that $\psi_0 = \pi$, $\theta_0 = \pi / 2$ for the steady rotations of the first type, and $\psi_0 = \pi / 2$, $\theta_0 = \pi / 4$ for the steady rotations of the second type. Other values of the angles ψ_0 and θ_0 satisfying the equations (3.2) and (3.4), can be reduced to the values quoted above by altering the direction of the axis of the associated coordinate system.

4. Stability of the steady rotation of the first type. We consider the stability of the steady rotations of the first type obtained above, limiting ourselves to the stability in the first approximation. Let ψ and θ

denote the deviations of the precession and nutation angles from their equilibrium values ψ_0 and θ_0. For a steady rotation of the first type the system or equations of perturbed motion, will be as follows in the first approximation:

$$\psi'' + 3(a-1)A_2\psi + \varepsilon 3(a-1)A_3 \sin \omega_z\tau\theta = 0 \qquad (4.1)$$
$$\theta'' + \theta + \varepsilon 3(a-1)A_3 \sin \omega_z\tau\psi = 0$$

From (4.1) we see that when $\varepsilon = 0$, stability will occur provided that $a > 1$. When $\varepsilon \neq 0$, instability caused by a parametric resonance becomes possible. Let $\omega_1 = \sqrt{3(a-1)A_2}$ and $\omega_2 = 1$ be the oscillation eigenfrequencies in the system (4.1) with $\varepsilon = 0$. Since for $\varepsilon = 0$ the system (4.1) has obviously a sign-definite energy integral, the system can become unstable for $\varepsilon \neq 0$ only near such values of the parameters for which the quantities $2\omega_1$ and $2\omega_2$, or $\omega_1 + \omega_2$, are multiples of ω_z [4].

It can be shown that the inequalities $1.252 < \omega_z < 2$ hold for all μ, hence it follows that the quantity $2\omega_2$ cannot be a multiple of ω_z. Restricting ourselves to the first order approximation in ε we find, that instability is possible in the case under consideration, if either $2\omega_1$, or $\omega_1 + \omega_2$ is equal to ω_z. The corresponding values of the parameter a are:

$$a_1 = \frac{13}{12}, \quad a_2 = \frac{4\omega_z^2 - 2\omega_z + 1}{3\omega_z^2} \qquad (4.2)$$

When $\varepsilon \neq 0$, the regions of instability in the a, ε -plane should emerge, generally speaking, from the points of the axis a corresponding to the values a_1 and a_2. The boundaries of these regions can be found by writing e.g. the system (4.1) in the Hamiltonian form and applying a canonical transformation which would eliminate from the Hamiltonian the nonresonant terms. A straightforward analysis of the resulting simplified Hamiltonian will then yield the region of stability and instability of the system (4.1). The computations become particularly simple when $\varepsilon = 0$ and the linear system of equations of perturbed motion is reduced to equations describing the oscillation of oscillators not coupled to each other. This is precisely what happens in the case of the equations (4.1).

Computations have shown that in the first approximation in ε the region of instability becomes apparent in the neighborhood of $a = a_2$, but not of $a = a_1$. Its boundaries are given by the equations

$$a = a_2 \pm \varepsilon \frac{(\omega_z - 1)^{5/2}}{3\omega_z^4} A_3 \qquad (4.3)$$

For the system Earth – Moon $A_3 = 15.8452$ and the boundaries (4.3) of the region of instability are

$$a = 1.065 \pm 0.284\varepsilon$$

5. **Stability of the steady rotations of the second type.** The linearized equations of perturbed motion for a steady rotation of the second type have the form

$$\psi'' + (\gamma + 1)\theta' - \gamma\psi - \varepsilon\gamma A_3 / A_2 \sin \omega_z\tau\psi = 0 \qquad (5.1)$$

$$\theta'' - {}^1/_2\,(\gamma + 1)\,\psi' - {}^1/_2\,(\gamma - 1)\,\theta - \varepsilon 2\gamma A_3 / A_2 \,\sin\, \omega_z \tau\theta = 0$$
$$\gamma = 3\,(a - 1)\,A_2$$

We shall first consider the stability of the system (5.1) for $\varepsilon = 0$. When $\varepsilon = 0$, the root of its characteristic equation can easily be shown to be purely imaginary only when the inequality $\gamma\,(\gamma - 1) > 0$ holds. The latter inequality provides the necessary conditions of stability of the system (5.1) for $\varepsilon = 0$. If $\gamma\,(\gamma - 1) \neq 2$, then this condition will also become sufficient, since in this case the frequencies ω_i $(i = 1, 2)$ of the linear oscillations will differ from each other. The frequencies satisfy the equation

$$2\,\omega^4 - (\gamma^2 - \gamma + 2)\,\omega^2 + \gamma\,(\gamma - 1) = 0 \qquad (5.2)$$

We shall assume that $\omega_1 > \omega_2 > 0$. Then

$$\omega_1 = 1,\ \omega_2 = [\gamma\,(\gamma - 1) / 2]^{1/2},\ 0 < \gamma\,(\gamma - 1) < 2 \qquad (5.3)$$
$$\omega_1 = [\gamma\,(\gamma - 1) / 2]^{1/2},\ \omega_2 = 1,\ \gamma\,(\gamma - 1) > 2$$

The regions of stability can also be conveniently described with help of the inertial parameter a of the rigid body. Using this approach we find, that we have stability when $\varepsilon = 0$, provided that a belongs either to region 1 consisting of two intervals

$$0 < a < 1 - 1 / (3\,A_2),\ 1 - 1 / (3\,A_2) < a < 1$$

or to region 2 also consisting of two intervals

$$1 + 1 / (3\,A_2) < a < 1 + 2 / (3\,A_2),\ \ 1 + 2 / (3\,A_2) < a < 2$$

We note that the values $a = 1 - 1 / (3\,A_2)$ and $a = 1 + 2 /(3A_2)$ of the parameter are excluded from our consideration, since they correspond to the case of identical frequencies $(\omega_1 = \omega_2 = 1)$.

Let us now assume that the parameter ε is not zero. To find the regions of instability is it convenient to write the equations (5.1) in the Hamiltonian form, choosing the canonical variables in such a manner that when $\varepsilon = 0$, then the Hamiltonian function becomes a sum of Hamiltonians of two independent oscillators with frequencies of ω_1 and ω_2. Assuming in this case

$$q_\psi = \psi,\ q_\theta = \theta$$
$$p_\psi = \psi' + {}^1/_2\,(\gamma + 1)\,\theta, \quad p_\theta = 2\,\theta' - {}^1/_2\,(\gamma + 1)\,\psi$$

and performing a canonical change of variables according to the algorithm given in [5],

$$q_\psi = -\,2\,(\gamma + 1)\,[\varkappa_1\omega_1 q_1 \pm \varkappa_2\omega_2 q_2] \qquad (5.4)$$
$$q_\theta = 2\,[(\gamma + \omega_1^2)\,\varkappa_1 p_1 + (\gamma + \omega_2^2)\,\varkappa_2 p_2]$$
$$p_\psi = (\gamma + 1)\,[(\gamma - \omega_1^2)\,\varkappa_1 p_1 + (\gamma - \omega_2^2)\,\varkappa_2 p_2]$$
$$p_\theta = [(\gamma - 1)^2 - 4\,\omega_1^2]\,\varkappa_1\omega_1 q_1 \pm [(\gamma - 1)^2 - 4\,\omega_2^2]\,\varkappa_2\omega_2 q_2$$
$$(\varkappa_i = \{4\,\omega_i \mid (\gamma + 1)\,[(\gamma + 2)\,\omega_i^2 - \gamma^2] \mid\}^{-1/2},\ \ i = 1, 2)$$

we obtain the Hamiltonian function of the linearized equations of perturbed motion in the form

$$
\begin{aligned}
H &= H^\circ + \varepsilon H^1 \qquad (5.5)\\
H^\circ &= {}^1/_2\, \omega_1 (q_1^2 + p_1^2) \pm {}^1/_2\, \omega_2 (q_2^2 + p_2^2)\\
H^1 &= -2\, \gamma A_3 / A_2 \,[(\gamma + 1)^2 \varkappa_1^2 \omega_1^2 q_1^2 + 4 (\gamma + \omega_1^2)^2 \varkappa_1^2 p_1^2 +\\
&\quad (\gamma + 1)^2 \varkappa_2^2 \omega_2^2 q_2^2 + 4 (\gamma + \omega_2^2)^2 \varkappa_2^2 p_2^2 +\\
&\quad 8 (\gamma + \omega_1^2)(\gamma + \omega_2^2)\, \varkappa_1 \varkappa_2 p_1 p_2 \pm\\
&\quad 2 (\gamma + 1)^2 \varkappa_1 \varkappa_2 \omega_1 \omega_2 q_1 q_2]\, \sin \omega_z \tau
\end{aligned}
$$

The upper sign in the formulas (5.4) and (5.5) refers to the region 1, and the lower sign to the region 2.

Next we consider the problem of parametric resonance for $\varepsilon \neq 0$. We deal first with region 1 where for $\varepsilon = 0$ we have a sign definite energy integral $H^\circ =$ const. In the first approximation in ε, an instability may occur near those values of a, for which one of the following resonance relationships holds: $2\omega_1 = \omega_z$, $2\omega_2 = \omega_z$ or $\omega_1 + \omega_2 = \omega_z$. An analysis carried out with the help of (5.2) has shown that the first of the above resonance relationships is impossible in region 1, while the second and third relationship are realized, respectively, for

$$a_3 = 1 + \frac{1 - [1 + 2\omega_z^2]^{1/2}}{6\omega_z^2}, \quad a_4 = 1 + \frac{1 - [1 + 8(\omega_z - 1)^2]^{1/2}}{6\omega_z^2}$$

The boundaries of the region of instability originating at the point $a = a_3$ are described, as shown by the computations, by

$$a = a_3 \pm \varepsilon \frac{|\gamma|\, |4(\gamma + \omega_2^2)^2 - (\gamma + 1)^2 \omega_2^2|\, \varkappa^2_2}{\omega_z^2\, |d\omega_2/da|} A_3 \qquad (5.6)$$

For the system Earth — Moon, the above relations become

$$a = 0.91 \pm 0.008\varepsilon$$

The region of instability originating at the point $a = a_4$, has the following boundaries:

$$a = a_4 \pm \varepsilon 2 \frac{|\gamma|\, |4(\gamma + \omega_1^2)(\gamma + \omega_2^2) - (\gamma + 1)^2 \omega_1 \omega_2|\, \varkappa_1 \varkappa_2}{\omega_z^2\, |d(\omega_1 + \omega_2)/da|} A_3 \qquad (5.7)$$

and for the system Earth — Moon these boundaries become

$$a = 0.925 \pm 0.37\varepsilon$$

Let us now consider region 2 where the integral $H^\circ =$ const is not sign definite when $\varepsilon = 0$. In the first approximation in ε an instability is possible near the values of a for which one of the following resonance relations holds: $2\omega_1 = \omega_z$, $2\omega_2 = \omega_z$, $\omega_1 - \omega_2 = \omega_z$. An analysis has shown that the first resonance in region 2 is impossible, while the second and third resonances are possible. The corresponding values of the parameter a are

$$a_5 = 1 + \frac{1 + [1 + 2\omega_z^2]^{1/2}}{6\omega_z^2}, \quad a_6 = 1 + \frac{1 + [1 + 8(\omega_z + 1)^2]^{1/2}}{6\omega_z^2}$$

The boundaries of the region of instability emerging from the point a_5 are given by (5.7) in which a_3 is replaced by a_5. For the system Earth —Moon these equations have the form

$$a = 1.194 \pm 7.164\varepsilon$$

The boundaries of instability near the resonance value $a = a_6$ are given by (5.7) in which a_4 and ω_2 are replaced by a_6 and $-\omega_2$ respectively. For the system Earth —Moon these boundaries are given by the equations

$$a = 1.467 \pm 3.095\varepsilon$$

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